6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

We do not have a function for h(x). But, a tangent line to a point on a curve has the same slope as that point on the curve. Let's rewrite the given tangent line equation in point-slope form.

$$y = 4 + \frac{2}{3}(x-2)$$

$$y - 4 = \frac{2}{3}(x-2)$$

Point: $(2,4)$

Slope: $\frac{2}{3}$

$$\boxed{h'(2) = \frac{2}{3}}$$

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

To take the derivative of a(x), we need need to use Product Rule and Power Rule.

$$a'(x) = 3x^3 \cdot h'(x) + h(x) \cdot 9x^2$$

$$a'(2) = 3(2)^3 \cdot h'(2) + h(2) \cdot 9(2)^2$$

$a'(2) = 3(8) \cdot \frac{2}{3} + 4 \cdot 9(4)$

Leave answer here on AP Test

$a'(2) = \cancel{24}^{8} \cdot \frac{2}{\cancel{3}} + 4 \cdot 36$

$a'(2) = 16 + 144$

$a'(2) = 160$

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(c) The function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

This problem takes some thinking.

Firstly, because $h(2) = 4$ then $\lim_{x \to 2} h(x) = 4$.

Secondly, because L'Hospital's Rule can be used to evaluate $\lim_{x \to 2} h(x)$ then both the numerator and denominator for $h(x)$ must equal 0.

$$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} \Rightarrow \frac{\lim_{x \to 2} x^2 - 4}{\lim_{x \to 2} 1 - (f(x))^3} \Rightarrow \frac{0}{0}$$

Now, we only are concerned with $f(x)$ which is in the denominator.

$$\lim_{x \to 2} 1 - (f(x))^3 = 0$$

$$1 - (f(x))^3 = 0$$

$$1 = (f(x))^3$$

$$1 = f(x)$$

$$\lim_{x \to 2} f(x) = 1$$

Since $f(x)$ is differentiable, and thus continuous, the $\lim_{x \to 2} f(x)$ must equal to the value of $f(2)$. $\therefore \boxed{f(2) = 1}$

Now, to find $f'(2)$ we must apply L'Hospital's Rule to the limit of $h(x)$.

$$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$$

Use Chain Rule to take the derivative of the denominator.

$$\lim_{x \to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)}$$

Again, this limit must be equal to 4 since $h(2) = 4$.

$$\frac{2(2)}{-3(f(2))^2 \cdot f'(2)} = 4$$

$f(2) = 1$

$$\frac{4}{-3(1)^2 \cdot f'(2)} = 4$$

$$\frac{4}{-3 \cdot f'(2)} = 4$$

$$\frac{4}{4} = -3 \cdot f'(2)$$

$$1 = -3 \cdot f'(2)$$

$$\boxed{-\frac{1}{3} = f'(2)}$$

$$\lim_{x \to 2} f'(x) = -\frac{1}{3}$$

Again, since f is twice-diffentiable

f'(x) is also continuous.

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

This is one of those common thought or theorem problems.

Because both g and h are differentiable and thus continuous, and $g(2) = h(2)$; by Squeeze Theorem, $g(2) = k(2) = h(2)$ and $k(2)$ is thus continuous.